A solution of the same problem by a somewhat simpler method might be of interest.

Let X be the average number of "rolls" needed further to decide a "point." Now we know that it is necessary to make at least one roll; and the probability of this, being certainty, is expressed by unity. If the "point" is not decided at the first roll (the probability of which is p or $1 - q$), the average number of rolls needed is still X. Therefore we arrive at the following equality,

$$X = 1 \times 1 + (1 - q)X$$

from which we find that $X = 1/q$.

This solution and that suggested by Mr. Brown are given by Louis Bachelier in his *Calcul des Probabilités*, volume 1, Paris, 1912, page 11.

III. Concerning the Teaching of Logarithms.

By A. F. Frumveller, Marquette University.

In this Monthly for September, 1919, Professor McClenon brought up the question of introducing logarithms by the historic method of geometric progression; he mentions as an alternative, the complete abandonment of all theory, and the laying down of mechanical rules of thumb for the student until the use of the tables has become automatic. The first or historic method has, it seems, been tried out successfully at the Hyde Park High School, Chicago; Mr. Josef Nyberg explained his proceedure in the October number of this Monthly for 1918, p. 337, but on reading his explanation it would seem as if the lack of directness of this approach would render it unsatisfactory in the hands of teachers less skillful and inspiring than Mr. Nyberg himself.

The late S. A. T. C. arrangement gave the present writer a splendid opportunity of trying out on a large scale the teaching of logarithms to students of average ability; and it so happened that the few days spent in this study afforded the only agreeable interlude in an experience that to most of us was like a nightmare. Our method was this; an equation like $y = x^2$ is written down; we define 2 to be the logarithm of y to the base x, and write $\log_x y = 2$ as the statement of this definition. This we call "translating the exponential equation to the logarithmic form," and the student is drilled at once in making translations of this kind by the dozens, till he can do it automatically; numerical cases like $9 = 3^2$ are next handled orally, with all possible modifications; *e.g.*, if 3 is the log of 8, what base am I using? etc. Many algebras are well supplied with such oral exercises; after a lively quiz of this sort, the students, *all* of them, had the idea of a logarithm firmly fixed in their minds. The rules for operating with logarithms gave no difficulty, since by definition a logarithm is merely another name for an exponent; fractional and negative exponents were then brought in and *translated* into logarithmic form, using always an exponential equation $y = x^{\pm n}$ as a starting point, and then employing numbers in place of y, x, and n, till the idea took root.

This plan worked excellently; the students learned rapidly how to operate this new notation; as regards the tables, they were merely told, that just as in banks the computation of interest is never done by the schoolboy's rules of arithmetic, but by means of tables which have been computed in advance, so in mathematics, the exponents of all possible numbers have been computed for the base 10 to save us trouble; the students are to feel free to use these tables. We always spoke of the tables as giving *exponents*; "look in the table of exponents" was one of our standard remarks.

Fractional exponents like 1/3 will be found in the tables in decimal notation, as 0.33333; to make this usage familiar, we had only to take our old equation $y = x^n$ and replace n by some terminating decimal like 0.25 or 0.125 to begin with, and then go back to the fractional form and interpret what we had. In regard to decimals, we defined that if a number like 7.25 is given, we call 7 the characteristic, and 25 the mantissa *of this number*. And right here we met the first and the only difficulty! Some of the students had used negative characteristics in their former work, writing for instance $-$ 3.93817 as the log of .008673; the idea was to get the sixth root of this number. On dividing by 6, we got $-$ .65636! What was the matter with this result? I recommended that we write $3.93817 - 6$ before division, which gives $0.65936 - 1$, correctly; but at once some one wanted to know why we could not write $7.93817 - 10$ just as well, the result being apparently altogether different? There was no need of assigning this topic for the next recitation! The way in which we finally threshed the matter out was this.

The decimal notation .008673 is merely a shorthand notation for the proper fraction $\frac{8,673}{1,000,000}$ whose log is $3.93817 - 6$; in every case therefore, the log consists of two parts, a positive part for the numerator, and a negative part for the denominator. In taking the sixth root, we must operate on both numerator and denominator; hence 6 is to be used as a divisor on both parts of the logarithm. Moreover there is not, and cannot be, any other correct form for the writing of the logarithm of a decimal; the log is essentially composed of two parts, the first of which is integral and decimal, the second integral only, since it represents some power of ten. Any other way of writing such logs is essentially wrong and misleading; I can transform my logarithm by adding to it zero in the shape of $m - m$; I can actually perform the subtraction indicated between the two parts and write $3.93817 - 6 = -2.06183$; but I cannot reasonably or intelligently keep the decimal unchanged and merely subtract the integers from one another, so as to have $3.93817 - 6 = -3.93817$! What would any one say if I were asked to subtract 4 from $3\frac{1}{7}$ and I were to write the result thus: $3.14285 - 4 = -1.14285$? Yet this foolishness is what many writers recommend, and no wonder the student is mystified; a number like 3.93817 is one definite mental object, and when a minus sign precedes it, this minus sign always refers to the entire number, and not to a certain percentage of the digits that compose it! Consider the following variety of ways for finding $\sqrt[6]{0.008673}$:

(1) $\dfrac{6)3.93817 - 6}{0.65636 - 1}$, (2) $\dfrac{6)57.93817 - 60}{9.65636 - 10}$,

$n = (4.5329)/10 = 0.45329;$ $n = (4532900000)/10{,}000{,}000{,}000 = 0.45329;$

(3) $\dfrac{6)7.93817 - 10}{1.32303 - 1.66666}$, (4) $\dfrac{6)2.93817 - 5.00000}{0.48969 - 0.83333}$,

$n = (2.104)/(4.641) = 0.4533;$ $n = (3.088)/(6.813) = 0.4533;$

(5) $\dfrac{6)-2.06183}{-0.34364}$, (6) $\dfrac{6)-3.93817}{-0.65636}$,

$n = 1/(2.206) = 0.4533;$ $m = 1/(4.5329) = 0.2206.$

The first two are the standard methods, but all are correct, except the last; here by a *mere accident,* the digits of the quotient turn out to be right, though the negative sign throws the number into the denominator; had we taken the seventh root, not even the digits would have been found, much less the proper sign!

After studying these results, the class voted unanimously that the use of negative characteristics was absurd and misleading; every one of these boys could use logarithms easily and correctly from that time forward. As an experiment in teaching logarithms, complete success was obtained, so that the writer sees no need of ever changing this method.

As a matter of curiosity, the question of settling under what circumstances the digits of the mantissa (in case no. 6 above) turn out correct was looked into; the reader may be interested in the answer. Let us use 0_k as a symbol for k zeros adjacent to one another in a given decimal number N; and let

$$\begin{aligned}\log N^x &= \log (0.0_k n_1 \cdots n_\lambda)^x \\ &= x[\log (n_1 \cdots n_\lambda) - \log (10^{k+\lambda})] \\ &= x[(\lambda - 1).s_1 \cdots s_5 - (k + \lambda)] \text{ as correctly written.}\end{aligned}$$

Written *wrongly,* by subtracting the integral parts of these numbers while leaving the decimal part untouched, we get

$$\log (N^x) = x[-(k+1)\cdot(s_1 \cdots s_5)].$$

Now let $x = 1/y$; then if $|k + 1| \equiv (\lambda - 1)$, (mod y), *i.e.*, if the quotients $(k + 1)/y$, $(\lambda - 1)/y$, have the same remainders, the mantissa $(s_1' \cdots s_5')$ in both cases will be the same.

IV. Relating to the Analytical Geometry of the Circle.

By Charles N. Schmall, New York.

E. H. Askwith in his *Analytical Geometry of the Conic Sections* (London, A. & C. Black, 1908), p. 78, § 90, says: "We could by analysis prove all the geometrical properties of the circle. It must not, however, be supposed that